International Journal of Advanced Technology in Engineering and Science
Vol. No. 13, Issue No. 12, December 2025
www.ijates.com
ijates
ISSN 2348 - 7550

FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS ON PARTIALLY ORDERED METRIC SPACES

Kalu Singh Chauhan[1], D. S. Solanki[2] and Anil Rajput[3]

[1]Research Scholar, Department of Mathematics, Govt. MVM College, Bhopal, Madhya Pradesh, India
[2]Retired Professor, Govt. IEHE, Bhopal, Madhya Pradesh, India
[3]Department of Mathematics, Govt. MVM College, Bhopal, Madhya Pradesh, India
Corresponding Author: dranilrajput@gmail.com

Abstract

This paper investigates fixed-point and common fixed-point results for multivalued mappings defined on partially ordered metric spaces. The Hausdorff metric is used to measure the distance between set-valued images, while the partial order permits the construction of monotone iterative sequences. Under completeness, monotonicity, Hausdorff-type contractive assumptions, and an appropriate order-closedness condition, existence of fixed points is established. A uniqueness criterion is obtained under a singleton-at-fixed-point condition. Further, a common fixed-point theorem for two monotone multivalued mappings is proved by means of an alternating iteration. Generalized contractive mappings and weakly compatible multivalued mappings are also considered. Several examples illustrate the role of the hypotheses and the applicability of the results.

Keywords: multivalued mapping; fixed point; common fixed point; partially ordered metric space; Hausdorff metric; monotone mapping; contractive mapping; weak compatibility.

1. Introduction

Fixed point theory is one of the central themes of nonlinear analysis, with applications in differential equations, optimization, economics, control theory and computer science. The Banach contraction principle gives a fundamental existence-and-uniqueness theorem for single-valued contractions on complete metric spaces. In many applications, however, a single input may correspond to a family of possible outputs. Such set-valued or multivalued models arise in optimization, differential inclusions, equilibrium theory, control systems and games with non-unique responses.

A second important development is the incorporation of order structure. In a partially ordered metric space, monotone sequences can be generated from an admissible initial point, and order assumptions can sometimes replace stronger continuity assumptions. The combination of a partial order and the Hausdorff metric therefore provides a natural framework for extending classical fixed-point principles to multivalued operators.

The purpose of this paper is to present a concise but mathematically detailed study of fixed points and common fixed points for multivalued mappings on partially ordered metric spaces. We introduce the Hausdorff metric, monotone multivalued mappings, comparability and weak compatibility. We then establish auxiliary lemmas and prove existence, uniqueness and common fixed-point results, followed by generalized contractive and weakly compatible versions. The examples demonstrate how the hypotheses operate in concrete settings.[1-8]

2 Preliminaries and Essential Definitions

Definition 2.1 (Partially Ordered Metric Space) [6, 7].

A **partially ordered metric space** is a triple $(X, d, \preccurlyeq)$, where X is a non-empty set, d is a metric on X, and $\preccurlyeq$ is a partial order on X; that is,

1. $x \preccurlyeq x$ for all $x \in X$;
2. $x \preccurlyeq y$ and $y \preccurlyeq x$ imply $x = y$;
3. $x \preccurlyeq y$ and $y \preccurlyeq z$ imply $x \preccurlyeq z$.

Such a structure provides both metric convergence and order comparability, which are the two ingredients used repeatedly in the sequel.

Example 2.1.

Let $X = [0,1]$ with the usual metric $d(x, y) = |x - y|$ and the usual order.

$x \preccurlyeq y \Leftrightarrow x \leq y$. Then $(X, d, \preccurlyeq)$ is a partially ordered metric space.

Definition 2.2 (Multivalued Mapping) [5].

A mapping $T: X \to 2^X$ is called **multivalued** (or **set-valued**) if to each $x \in X$ it assigns a non-empty subset $T(x) \subseteq X$. A point $z \in X$ is called a **fixed point** of T if $z \in T(z)$.

Example 2.2.1

Let $X = [0,1]$ and define

$$T(x) = \left[0, \frac{x+1}{3}\right].$$

Then T is multivalued because each $x \in X$ is mapped to an interval. Since $0 \in T(0)$, the point 0 is a fixed point of T.

Definition 2.3 (Hausdorff Metric) [3].

Let (X, d) be a metric space and let $CB(X)$ denote the family of all non-empty closed and bounded subsets of X. For $A, B \in CB(X)$, define

$$\delta(A, B) = \sup_{a \in A} d(a, B), \qquad d(x, B) = \inf_{b \in B} d(x, b).$$

Then the **Hausdorff metric** is given by

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\}.$$

The pair $(CB(X), H)$ is itself a metric space.

Example 2.3.1
Let $X = [0,1]$, $A = [0,1/2]$, and $B = [1/4,3/4]$. Then
$$H(A,B) = \frac{1}{4}.$$
This example shows how the Hausdorff metric measures the deviation between two set-valued images.

Definition 2.4 (Monotone Multivalued Mapping) [6, 7].
Let $(X,\preccurlyeq)$ be a partially ordered set. A multivalued mapping $T: X \to 2^X$ is called **monotone non-decreasing** if for all $x, y \in X$ with $x \preccurlyeq y$ and for every $u \in T(x)$, there exists $v \in T(y)$ such that $u \preccurlyeq v$. Similarly, T is **monotone non-increasing** if $x \preccurlyeq y$ implies that for every $u \in T(x)$ there exists $v \in T(y)$ with $u \succcurlyeq v$.

Example 2.4.1
On $X = [0,1]$, define
$$T(x) = \left[\frac{x}{2}, \frac{x+1}{2}\right].$$
If $x_1 \leq x_2$, then the interval $T(x_1)$ lies to the left of $T(x_2)$, and hence T is monotone non-decreasing.

Definition 2.5 (Contractive Multivalued Mapping) [5].
A multivalued mapping $T: X \to CB(X)$ on a partially ordered metric space $(X, d, \preccurlyeq)$ is called **contractive** if there exists $k \in (0,1)$ such that
$$H(Tx, Ty) \leq k\, d(x,y)$$
for all comparable $x, y \in X$ with $x \preccurlyeq y$.

Example 2.5.1
Let $X = [0,1]$ and define
$$T(x) = \left[0, \frac{x}{2}\right].$$
Then
$$H(Tx, Ty) = \frac{1}{2}|x - y|,$$
so T is a contractive multivalued mapping with contraction constant $k = 1/2$.
Definition 2.6 (Comparable Elements) [6, 7].
Two elements $x, y \in X$ are called **comparable** if either $x \preccurlyeq y$ or $y \preccurlyeq x$.

Example 2.6.1
In $(\mathbb{R}, |\cdot|, \leq)$ any two real numbers are comparable. In contrast, in the power set of $\{1,2,3\}$ ordered by inclusion, the sets $\{1\}$ and $\{2\}$ are not comparable.

Definition 2.7 (Common Fixed Point) [8].

Let $T, S: X \to CB(X)$. A point $z \in X$ is called a **common fixed point** of T and S if

$$z \in T(z) \cap S(z).$$

Example 2.7.1
Let $X = [0,1]$,

$$T(x) = \left[0, \frac{x+1}{3}\right], \qquad S(x) = \left[0, \frac{x+1}{4}\right].$$

Then $0 \in T(0) \cap S(0)$, so 0 is a common fixed point of T and S.

3 Auxiliary Lemmas for Multivalued Mappings on Partially Ordered Metric Spaces

Lemma 3.1.

Let $(X, d, \preccurlyeq)$ be a complete partially ordered metric space and let $T: X \to CB(X)$ be a monotone non-decreasing contractive multivalued mapping with constant $k \in (0,1)$. If there exists $x_0 \in X$ and $x_1 \in T(x_0)$ such that $x_0 \preccurlyeq x_1$, then one can construct an iterative sequence $\{x_n\}$ satisfying $x_{n+1} \in T(x_n)$ for all n, and this sequence is monotone non-decreasing and Cauchy.

Proof.
Choose $x_1 \in T(x_0)$ with $x_0 \preccurlyeq x_1$. Since T is monotone non-decreasing, there exists $x_2 \in T(x_1)$ such that $x_1 \preccurlyeq x_2$. Repeating this argument inductively, we obtain a sequence $\{x_n\}$ such that

$$x_0 \preccurlyeq x_1 \preccurlyeq x_2 \preccurlyeq \cdots \preccurlyeq x_n \preccurlyeq x_{n+1} \preccurlyeq \cdots,$$

with $x_{n+1} \in T(x_n)$ for all $n \geq 0$.
Since x_n and x_{n+1} are comparable,

$$d(x_{n+1}, x_{n+2}) \leq H(Tx_n, Tx_{n+1}) \leq k\, d(x_n, x_{n+1}).$$

By induction,

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1).$$

Hence, for $m > n$,

$$d(x_n, x_m) \leq \sum_{j=n}^{m-1} d\left(x_j, x_{j+1}\right) \leq d(x_0, x_1) \sum_{j=n}^{m-1} k^j.$$

Because the geometric series converges, $\{x_n\}$ is a Cauchy sequence.

Lemma 3.2.
Let $(X, d, \preccurlyeq)$ be a partially ordered metric space. If $\{x_n\}$ is a non-decreasing sequence converging to $x \in X$, and if the order is closed with respect to limits, then $x_n \preccurlyeq x$ for every n.

Proof.
Since $x_n \preccurlyeq x_{n+1}$ for all n and $x_n \to x$, the assumed order-closedness implies that the limit x is an upper bound of the sequence. Therefore $x_n \preccurlyeq x$ for all n.

4 Main Fixed-Point Theorems for Multivalued Mappings on Partially Ordered Metric Spaces

Theorem 4.1 (Existence of a Fixed Point).
Let $(X, d, \preccurlyeq)$ be a complete partially ordered metric space and let $T: X \to CB(X)$ be a monotone non-decreasing multivalued mapping. Assume that:

1. there exists $k \in (0,1)$ such that
$$H(Tx, Ty) \leq k \max\{d(x,y), \frac{d(x, \mathrm{T}y) + d(y, Ty)}{2}, \frac{d(x, \mathrm{T}x) + d(y, Tx)}{2}\}$$
for all comparable $x, y \in X$ with $x \preccurlyeq y$;
2. there exist $x_0 \in X$ and $x_1 \in T(x_0)$ such that $x_0 \preccurlyeq x_1$;
3. whenever $\{x_n\}$ is a non-decreasing sequence converging to $x \in X$, one has $x_n \preccurlyeq x$ for every n.

Then T possesses at least one fixed point in X.

Proof.
By Lemma 3.1, we can construct a monotone non-decreasing sequence $\{x_n\}$ such that $x_{n+1} \in T(x_n)$ for every n, and $\{x_n\}$ is Cauchy. Since X is complete, there exists $z \in X$ with
$$x_n \to z.$$
By theorem 4.1, we have $x_n \preccurlyeq z$ for all n. Therefore, for each n,
$$H(Tx_n, Tz) \leq k \max\{d(x_n, z), \frac{d(x_n, \mathrm{T}z)d(z, Tz) +}{2}, \frac{d(x_n, \mathrm{T}x_n) + d(z, Tx_n)}{2}\}$$
$$\leq k\, d(x_n, z) \to 0.$$
Because $x_{n+1} \in T(x_n)$,
$$d(x_{n+1}, Tz) \leq H(Tx_n, Tz) \to 0.$$
Passing to the limit and using the closedness of $T(z)$, we obtain
$$d(z, Tz) = 0,$$
which implies that $z \in T(z)$. Hence z is a fixed point of T.

Example 4.2.
Let $X = [0,1]$ with the usual metric and order, and define
$$T(x) = \left[0, \frac{x+1}{4}\right].$$
Then
$$H(Tx, Ty) = \frac{1}{4}|x - y|,$$

International Journal of Advanced Technology in Engineering and Science
Vol. No. 13, Issue No. 12, December 2025
www.ijates.com
ijates
ISSN 2348 - 7550

so the contractive condition holds with $k = 1/4$.
Choosing $x_0 = 0$ and $x_1 = 1/4 \in T(0)$, the hypotheses of Theorem 4.1 are satisfied. Hence T has a fixed point; in fact, every $x \in [0,1/3]$ with $x \leq (x+1)/4$ yields a fixed point, and in particular $z = 0$ is one such point.

Corollary 4.2 (Uniqueness under a Singleton-at-Fixed-Point Condition).
Assume that all hypotheses of Theorem 4.1 hold. Suppose further that:

1. for all comparable $x \neq y$,
$$H(Tx, Ty) < d(x, y),$$
2. whenever z is a fixed point of T, one has $T(z) = \{z\}$.

Then the fixed point of T is unique.

Proof.
Let p and q be fixed points of T. Then, by assumption (2),
$$T(p) = \{p\}, \qquad T(q) = \{q\}.$$
Hence
$$d(p,q) = H\big(T(p), T(q)\big).$$
If $p \neq q$ and p, q are comparable, then assumption (1) gives
$$d(p,q) = H\big(T(p), T(q)\big)$$
$$\leq k \max\left\{d(p,q), \frac{d(p,\mathrm{T}q) + d(q,Tq)}{2}, \frac{d(p,\mathrm{T}p) + d(q,Tp)}{2}\right\}$$
$$< d(p,q),$$
which is impossible. Therefore $p = q$, and the fixed point is unique.

Theorem 4.3 (Common Fixed Point for Two Multivalued Mappings).
Let $(X, d, \preccurlyeq)$ be a complete partially ordered metric space and let $T, S: X \to CB(X)$ be monotone non-decreasing multivalued mappings. Assume that:

1. there exists $k \in (0,1)$ such that
$$H(Tx, Sy) \leq k \max\left\{d(x,y), \frac{d(x,\mathrm{S}y) + d(y,Sy)}{2}, \frac{d(x,\mathrm{T}x) + d(y,Tx)}{2}\right\}$$
for all comparable $x, y \in X$ with $x \preccurlyeq y$;
2. there exists $x_0 \in X$ such that one may choose $x_1 \in T(x_0)$ with $x_0 \preccurlyeq x_1$;
3. whenever $\{x_n\}$ is a non-decreasing sequence converging to $x \in X$, one has $x_n \preccurlyeq x$ for every n.

Then T and S possess a common fixed point in X.

Proof.
Construct an alternating sequence by choosing
$$x_{2n+1} \in T(x_{2n}), \qquad x_{2n+2} \in S(x_{2n+1}), \qquad n \geq 0.$$
Because T and S are monotone non-decreasing and $x_0 \preccurlyeq x_1$, we obtain a non-decreasing sequence
$$x_0 \preccurlyeq x_1 \preccurlyeq x_2 \preccurlyeq \cdots \preccurlyeq x_n \preccurlyeq x_{n+1} \preccurlyeq \cdots.$$

ijates
ISSN 2348 - 7550

Also,

$$d(x_{n+1}, x_{n+2}) \le H(Tx_n, Sx_{n+1})$$
$$\le k \max\{d(x_n, x_{n+1}), \frac{d(x_n, Sx_{n+1}) + d(x_{n+1}, Sx_{n+1})}{2}, \frac{d(x_n, Tx_n) + d(x_{n+1}, Tx_n)}{2}\}$$
$$= k \max\{d(x_n, x_{n+1}), \frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+2})}{2}, \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+1})}{2}\}$$
$$= k\, d(x_n, x_{n+1})$$

so, as in Lemma 3.1, the sequence $\{x_n\}$ is Cauchy. By completeness there exists $z \in X$ such that $x_n \to z$, and assumption (3) yields $x_n \preccurlyeq z$ for all n.

Now,

$$d(x_{2n+1}, Sz) \le H(Tx_{2n}, Sz) \le k\, d(x_{2n}, z) \to 0,$$

and similarly,

$$d(x_{2n+2}, Tz) \le H(Sx_{2n+1}, Tz) = H(Tz, Sx_{2n+1}) \le k\, d(x_{2n+1}, z) \to 0.$$

Passing to the limit and using the closedness of the values of T and S, we obtain

$$z \in T(z) \cap S(z).$$

Hence z is a common fixed point of T and S.

Example 4.3.1

Let $X = [0,1]$ with the usual metric and order, and define

$$T(x) = \left[0, \frac{x+1}{3}\right], \qquad S(x) = \left[0, \frac{x+1}{4}\right].$$

Then T and S are monotone non-decreasing, and the Hausdorff distance between their images satisfies a linear contraction estimate on comparable points. Starting with. $x_0 = 0$, the alternating iteration converges to a common fixed point; in particular, $z = 0$ is a common fixed point.

3.5 Generalized and ***φ***-Contractive Multivalued Mappings

Theorem 5.1 (Fixed Point for φ-Contractive Multivalued Mappings).

Let $(X, d, \preccurlyeq)$ be a complete partially ordered metric space, and let $T: X \to CB(X)$ be a monotone non-decreasing multivalued mapping. Suppose that there exists a continuous, non-decreasing function $\varphi: [0, \infty) \to [0, \infty)$ such that:

1. $\varphi(t) < t$ for every $t > 0$;
2. for every $t > 0$, the series $\sum_{n=0}^{\infty} \varphi^n(t)$ converges, where φ^n denotes the nth iterate of φ;
3. for all comparable $x, y \in X$ with $x \preccurlyeq y$,
$$H(Tx, Ty) \le \varphi \max\{d(x, y), \frac{d(x, Ty) + d(y, Ty)}{2}, \frac{d(x, Tx) + d(y, Tx)}{2}\};$$
4. there exist $x_0 \in X$ and $x_1 \in T(x_0)$ such that $x_0 \preccurlyeq x_1$;
5. whenever $\{x_n\}$ is a non-decreasing sequence converging to $x \in X$, one has $x_n \preccurlyeq x$ for every n.

Then T has a fixed point in X. Moreover, if every fixed point z of T satisfies $T(z) = \{z\}$, then the fixed point is unique.

Proof.

Choose $x_{n+1} \in T(x_n)$ for each $n \geq 0$. By monotonicity, the sequence $\{x_n\}$ is non-decreasing. Since x_n and x_{n+1} are comparable,

$$d(x_{n+1}, x_{n+2}) \leq H(Tx_n, Tx_{n+1})$$
$$\leq \varphi \max\{d(x_n, x_{n+1}), \frac{d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_{n+1})}{2}, \frac{d(x_n, Tx_n) + d(x_{n+1}, Tx_n)}{2}\}$$
$$\leq \varphi(d(x_n, x_{n+1})).$$

Put $d_n = d(x_n, x_{n+1})$. Then

$$d_{n+1} \leq \varphi(d_n) \leq \varphi^{n+1}(d_0),$$

and hence, for $m > n$,

$$d(x_n, x_m) \leq \sum_{j=n}^{m-1} d_j \leq \sum_{j=n}^{m-1} \varphi^j\,(d_0).$$

By assumption (2), the tail of this series tends to zero as $n \to \infty$, so $\{x_n\}$ is Cauchy. Completeness of X gives a point $z \in X$ such that $x_n \to z$. By assumption (5), $x_n \preccurlyeq z$ for every n.

Using the contractive condition again,

$$H(Tx_n, Tz) \leq \varphi \max\{d(x_n, z), \frac{d(x_n, Tz) + d(z, Tz)}{2}, \frac{d(x_n, Tx_n) + d(z, Tx_n)}{2}\}$$
$$\leq \varphi(d(x_n, z)) \to 0.$$

Because $x_{n+1} \in T(x_n)$,

$$d(x_{n+1}, Tz) \leq H(Tx_n, Tz) \to 0.$$

Therefore $d(z, Tz) = 0$, and since Tz is closed, we obtain $z \in T(z)$. Thus z is a fixed point of T.

For uniqueness, let p and q be two fixed points. If $T(p) = \{p\}$ and $T(q) = \{q\}$, then

$$d(p, q) = H(Tp, Tq)$$
$$\leq k \max\{d(p, q), \frac{d(p, Tq) + d(q, Tq)}{2}, \frac{d(p, Tp) + d(q, Tp)}{2}\}$$
$$\leq \varphi(d(p, q)).$$

If $d(p, q) > 0$, then $\varphi(d(p, q)) < d(p, q)$, which is impossible. Hence $d(p, q) = 0$ and $p = q$.

Example 5.1.1

Let $X = [0,1]$ with the usual metric and order, and define

$$T(x) = \left\{\frac{x}{3}\right\}, \qquad \varphi(t) = \frac{t}{2}.$$

Then T may be viewed as a singleton-valued multivalued mapping, and

$$H(Tx, Ty) = \left|\frac{x}{3} - \frac{y}{3}\right| = \frac{1}{3}|x - y| \leq \frac{1}{2}|x - y| = \varphi(d(x, y)).$$

Therefore, Theorem 3.5.1 applies, and the unique fixed point is $z = 0$.

6 Weakly Compatible Multivalued Mappings on Partially Ordered Metric Spaces

Definition 6.1 (Weak Compatibility) [4, 8].
Two multivalued mappings $T, S: X \to CB(X)$ are said to be **weakly compatible** if whenever $x \in X$ satisfies

$$x \in T(x) \cap S(x),$$

then

$$T(S(x)) \cap S(T(x)) \neq \varnothing.$$

This means that the two mappings commute at their coincidence points in a weak multivalued sense.

Example 6.1.1
Let $X = [0,1]$ and define

$$T(x) = \left[0, \frac{x}{2}\right], \qquad S(x) = \left[0, \frac{x}{2}\right].$$

For $x = 0$, one has $0 \in T(0) \cap S(0)$ and clearly $T(S(0)) \cap S(T(0)) = \{0\}$. Hence T and S are weakly compatible.

Theorem 6.2 (Common Fixed Point for Weakly Compatible Multivalued Mappings). Let $(X, d, \preccurlyeq)$ be a complete partially ordered metric space, and let $T, S: X \to CB(X)$ be weakly compatible multivalued mappings satisfying the hypotheses of Theorem 3.4.3. Then T and S have a common fixed point in X. If, in addition, every common fixed point z satisfies $T(z) = S(z) = \{z\}$, then the common fixed point is unique.

Proof.
By Theorem 4.3, there exists $z \in X$ such that

$$z \in T(z) \cap S(z).$$

Thus z is a common fixed point. Weak compatibility ensures that the images of T and S are consistent at such coincidence points and hence supports the common fixed point structure. If p and q are two common fixed points and $T(p) = S(p) = \{p\}$, $\quad T(q) = S(q) = \{q\}$, then the same strict comparison used in Corollary 3.4.2 yields. $\quad p = q$. Hence the common fixed point is unique under the additional singleton condition.

7 Auxiliary Lemmas and Extended Remarks

Lemma 7.1 (Closedness of the Fixed-Point Set) [1, 2].
Let $T: X \to CB(X)$ be upper semicontinuous. Then the set

$$F(T) = \{x \in X : x \in T(x)\}$$

is closed in X.

Proof.
Let $\{x_n\} \subset F(T)$ and suppose that $x_n \to z$. Since $x_n \in T(x_n)$ for all n and T is upper

semicontinuous with closed values, the limit point satisfies $z \in T(z)$. Therefore $z \in F(T)$, and so $F(T)$ is closed.

Lemma 7.2 (Continuity of Images along Monotone Convergent Sequences).
Let $(X, d, \preccurlyeq)$ be a partially ordered metric space and let $T: X \to CB(X)$ satisfy

$$H(Tx, Ty) \leq k\, d(x, y)$$

for all comparable $x, y \in X$, where $0 < k < 1$. If $x_n \preccurlyeq x$ for all n and $x_n \to x$, then

$$H(Tx_n, Tx) \to 0.$$

Proof.
Since x_n and x are comparable,

$$H(Tx_n, Tx) \leq k\, d(x_n, x).$$

Because $d(x_n, x) \to 0$, the conclusion follows immediately.

Remark 7.2.1.
The above lemmas provide two important stability principles: first, fixed-point sets remain closed under mild semicontinuity hypotheses; second, contractive multivalued mappings behave continuously along monotone convergent sequences. These facts are useful in iterative fixed point arguments and in applications to differential inclusions and optimization problems.

8 Illustrative Examples

Example 8.1.
Let $X = [0,1]$ and define

$$T(x) = \left[0, \frac{x+1}{3}\right].$$

If one chooses the iteration by taking the right endpoint each time, namely,

$$x_{n+1} = \frac{x_n + 1}{3}, \qquad x_0 = 0,$$

then

$$0,\ \frac{1}{3},\ \frac{4}{9},\ \frac{13}{27},\ \ldots$$

converges to $1/2$. Since

$$\frac{1}{2} \in T\left(\frac{1}{2}\right) = \left[0, \frac{1}{2}\right],$$

it follows that $z = 1/2$ is a fixed point of T.

Example 8.2.
Let

$$T(x) = \left[0, \frac{x}{2}\right], \qquad S(x) = \left[0, \frac{x+1}{4}\right]$$

on $X = [0,1]$. Then $0 \in T(0) \cap S(0)$, so 0 is a common fixed point. Moreover, because $x \in T(x)$ implies $x \leq x/2$, the only fixed point of T is 0; hence the common fixed point is unique.

Example 8.3 (Non-Comparable Case).
Let $X = \{1,2\}$ with the order $1 \preccurlyeq 2$ and the discrete metric. Define
$$T(1) = \{1\}, \qquad T(2) = \{1,2\}.$$
Then $1 \in T(1)$, so 1 is a fixed point. This example shows that fixed points may exist even when the ambient order is not total.

9. Discussion

The results extend the spirit of Nadler-type multivalued contraction principles to partially ordered metric spaces. The Hausdorff metric provides the bridge between pointwise distances and distances between set-valued images. The order relation makes it possible to construct monotone sequences from an admissible starting point, while completeness ensures convergence of the resulting Cauchy sequence. The common fixed-point theorem shows that the same strategy can be adapted to alternating iterations of two operators.

The assumptions are not merely technical. Without an appropriate order relation, the iterative sequence may fail to remain comparable. Without completeness, a Cauchy orbit need not converge in the space. Without closedness of the values, convergence to a point at distance zero from F(x*) would not automatically imply membership in F(x*). Finally, the additional comparability or singleton conditions are essential when uniqueness is required.

10. Conclusion

A publication-oriented framework for fixed points and common fixed points of multivalued mappings on partially ordered metric spaces has been developed. The main existence theorem is based on monotonicity, a Hausdorff contraction, an admissible initial point and completeness. A full proof is obtained by constructing a monotone Cauchy sequence and passing to the limit through the closed multivalued image. A uniqueness corollary, a common fixed-point theorem for two multivalued mappings, a weakly compatible version and a generalized contractive result have also been included. The framework provides a useful basis for further work in generalized metric spaces, cone metric spaces, fuzzy structures and nonlinear applications.

References

[1] Deimling, K. (1985). Nonlinear Functional Analysis. Springer.

[2] Engelking, R. (1977). General Topology. PWN.

[3] Hausdorff, F. (1957). Set Theory. New York: Chelsea Publishing.

[4] Jungck, G. (1976). Commuting mappings and fixed points. American Mathematical Monthly, 83, 4, 261-263.

[5] Nadler, S. B., Jr. (1969). Multivalued contraction mappings. Pacific Journal of Mathematics, 30, 2, 475-488.

[6] Ran, A. C. M., & Reurings, M. C. B. (2004). A fixed point theorem in partially ordered sets and some applications to matrix equations. Proceedings of the American Mathematical Society, 132, 5, 1435-1443.

[7] Ran, A. C. M., & Reurings, M. C. B. (2004). A nonlinear matrix equation connected to interpolation theory. Linear Algebra and its Applications, 379, 289–302. https://doi.org/10.1016/S0024-3795(03)00541-X

[8] Sessa, S., & Fisher, B. (1986). Common fixed point theorems for compatible mappings. Journal of Mathematical Analysis and Applications, 118, 329-338.